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THE FORMULATION OF LINEARIZED BOUNDARY INTEGRAL EQUATIONS OF THE ANISOTROPIC THEORY OF ELASTICITY AND THEIR APPLICATION IN GEOMETRICAL INVERSE PROBLEMS†

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An elastic bounded anisotropic solid with an elastic inclusion is considered. An oscillating source acts on part of the boundary of the solid and excites oscillations in it. Zero displacements are specified on the other part of the solid and zero forces on the remaining part. A variation in the shape of the surface of the solid and of the inclusion of continuous curvature is introduced and the problem of the theory of elasticity with respect to this variation is linearized. An algorithm for constructing integral representations for such linearized problems is described. The limiting properties of the linearized operators are investigated and special boundary integral equations of the anisotropic theory of elasticity are formulated, which relate the variations of the boundary strain and stress fields with the variations in the shape of the boundary surface. Examples are given of applications of these equations in geometrical inverse problems in which it is required to establish the unknown part of the body boundary or the shape of an elastic inclusion on the basis of information on the wave field on the part of the body surface accessible for observation. © 1998 Elsevier Science Ltd. All rights reserved.

The problem of analysing the effect of small changes in the shape of the boundary on the wave fields in an elastic medium leads to the formulation of special boundary integral equations of the theory of elasticity [1–5]. The problem that is inverse to the problem of analysing the effect is to refine the a priori information on the shape of the unknown part of the boundary surface on the basis of data on the wave field and the part of the boundary of the medium accessible to observations (measurements), and is called the linearized inverse problem To solve it one needs to solve the direct problem using a system of classical boundary integral equations [6–8] and to change it to a system of special boundary integral equations. An iterative procedure of successive refinement of the shape of the required boundary enables non-linear inverse problems to be solved [2–5].

Suppose an elastic medium occupies a bounded region W, and an inclusion occupies a region $V^{(1)} \subset W$. The boundaries of the solid $\Gamma^+ = \partial W$ and of the inclusion $\Gamma^- = \partial W$ will be assumed to be surfaces of continuous curvature. We will assume that the inclusion $V^{(1)}$ and the external medium $V^{(0)} = W\backslash V^{(1)}$ are uniform, and each of them is characterized by a tensor $c^{(m)}$ of elasticity constants and a density $\rho^{(m)}(m = 0, 1)$ (see Fig. 1). The superscript $m = 1$ indicates that the quantity belongs to the medium of the inclusion, while the superscript $m = 0$ indicates that it belongs to the external medium. A time-varying source $p^*e^{-i\omega t}$, which excites oscillations in the elastic medium, acts on the part of the boundary of the solid $\Gamma^* \subset \Gamma^+$. We divide the remaining part of the boundary $\Gamma^+\backslash\Gamma^-$ into two parts: Γ_p^+ and Γ_u^+. Suppose that the boundary on the part Γ_p^+ is stress-free, while on Γ_u^+ the displacement field is zero. We will assume that steady oscillation conditions exist in the elastic medium.

After separating out the time factor $e^{-i\omega t}$ the above boundary-value problem is described by the equations of motion

$$\nabla\cdot\sigma^{(m)}+\rho^{(m)}\omega^2u^{(m)}=0,\quad x\in V^{(m)},\quad m=0,\ 1 \tag{1}$$

the constitutive relations

$$\sigma^{(m)}=c^{(m)}\odot\nabla u^{(m)},\quad x\in V^{(m)},\quad m=0,\ 1 \tag{2}$$

the boundary conditions

$$p^{(0)}=p^*;\quad x\in\Gamma^*;\quad p^{(0)}=0,\quad x\in\Gamma_p^+;\quad u^{(0)}=0,\quad x\in\Gamma_u^+ \tag{3}$$

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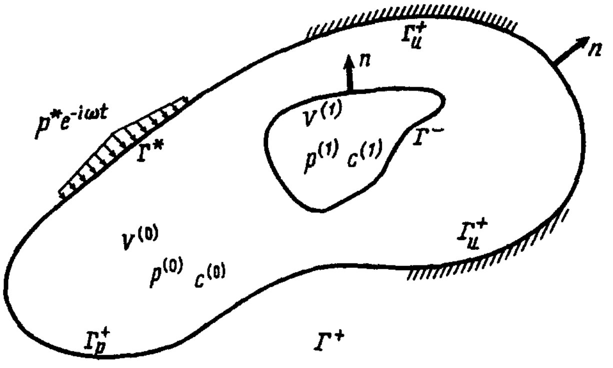

Fig. 1.

and the matching conditions at the inclusion boundary

$$u^{(0)} = u^{(1)}, \quad p^{(0)} = p^{(1)}, \quad x \in \Gamma^- \tag{4}$$

In the case of a cavity or an absolutely rigid inclusion it is sufficient to postulate a single boundary condition on Γ^{-1}

$$u^{(0)} = 0, \quad x \in \Gamma^- \quad \text{(an absolutely rigid inclusion)}$$
$$p^{(0)} = 0, \quad x \in \Gamma^- \quad \text{(a cavity)}$$

Here σ is the stress tensor, u is the displacement vector, and $p = \sigma \cdot n$ is the traction vector on an area with normal n. We will also assume that there is friction in the medium, proportional to the velocity. In this case it is sufficient to replace ω by ω_ε in the equations of motion, where $\omega_\varepsilon = \omega + i\varepsilon$, $\varepsilon > 0$.

Using the fundamental solutions U_r [9] of Eqs (1) and (2)

$$\nabla \cdot (c^{(m)} \odot \nabla U_r^{(m)}) + \rho^{(m)} \omega^2 U_r^{(m)} = -e_r \delta(x, \xi) \tag{5}$$
$$x, \ \xi \in \mathbf{R}^3, \quad r = 1, 2, 3$$

where e_r is the vector of a Cartesian basis, and $\delta(x, \xi)$ is the three-dimensional Dirac delta function, the integral representations for the displacement vector in terms of the wave fields on Γ^+ and Γ^- can be written in the form

$$u^{(m)}(\xi) = \mathbf{S}^{(m)}[u^{(m)}, p^{(m)}, \Gamma^{(m)}, \xi] \quad \xi \in V^{(m)}, \quad m = 0, 1$$
$$\Gamma^{(0)} = \partial V^{(0)} = \Gamma^+ \cup \Gamma^-, \quad \Gamma^{(1)} = \partial V^{(1)} = \Gamma^-$$
$$\mathbf{S}^{(m)} = (\mathbf{S}_1^{(m)}, \mathbf{S}_2^{(m)}, \mathbf{S}_3^{(m)})$$
$$\mathbf{S}_r^{(m)}[u, p, \Gamma, \xi] = \int_\Gamma (p(x) \cdot U_r^{(m)}(x, \xi) - u(x) \cdot P_r^{(m)}(x, \xi)) d\Gamma_x$$

$$P_r^{(m)}(x, \xi) = \Sigma_r^{(m)}(x, \xi) \cdot n(x), \quad \Sigma_r^{(m)}(x, \xi) = c^{(m)} \odot \nabla U_r^{(m)}$$

where $S^{(m)}$ is the Somigliana operator and n is the outward normal to Γ.

We will introduce a variation of the shape of the surface of continuous curvature Γ as a scalar function $\nu(x), x \in \Gamma, \nu \in C^1(\Gamma)$, which satisfies the following conditions

$$\|\nu,_s\| \ll 1, \quad \|\varkappa\nu\| \ll 1, \quad \|k\nu\| \ll 1 \tag{6}$$

Here $\|\cdot\| = \max_\Gamma |\cdot|$, $(\,)_{,s} = s \cdot \nabla(\,)$, s is any unit vector in the tangential plane to Γ, $\varkappa$ is the principal curvature of maximum modulus, and k is the maximum wave number.

We will specify the surface $\tilde{\Gamma}^-$ in a system of coordinates connected with Γ^- by means of the shape variation function ν

$$\tilde{r} = r + d, \quad d = \nu n, \quad x \in \Gamma^- \tag{7}$$

where r and $\tilde{r}$ are the radius vectors of the surfaces Γ^- and $\tilde{\Gamma}^-$. Similarly we introduce a surface $\tilde{\Gamma}^+$ and additionally require that $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$ should be simple surfaces and should not have common points.

Consider boundary-value problem (1)–(4) in regions bounded by the surfaces $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$. All the quantities relating to this boundary-value problem will be denoted by tilde. We linearized the difference

$$\tilde{u}^{(m)}(\xi) - u^{(m)}(\xi) = \mathbf{S}^{(m)}[\tilde{u}^{(m)}, \tilde{p}^{(m)}, \tilde{\Gamma}^{(m)}, \xi] - \mathbf{S}^{(m)}[u^{(m)}, p^{(m)}, \Gamma^{(m)}, \xi]$$

$$\xi \in \tilde{V}^{(m)} \cap V^{(m)}, \quad m = 0, 1 \tag{8}$$

The superscript m will henceforth be omitted.

Suppose (α, β) is an orthogonal system of coordinates on Γ. We will introduce the following notation: $(\ )_\alpha = \partial(\ )/\partial\alpha$, $(\ )_\beta = \partial(\ )/\partial\beta$ are orthogonal vectors in the tangential plane and $a = e_\alpha/|r_\alpha|$, $b = r_\beta/|r_\beta|$ is an orthonormalized basis in the tangential plane. The Jacobian of the conversion of an element of the surface $d\Gamma$ into an element of the surface $d\tilde{\Gamma}$ has the form

$$J = [(1 - \varkappa_1\nu)^2(1 - \varkappa_2\nu)^2 + \nu^2,_a + \nu^2,_b + 2\nu(\nu,_a N - 2\nu,_a \nu,_b M + \nu,_b L) +$$

$$+ \nu^2[\nu^2,_a(N^2 + M^2) + 2\nu,_a \nu,_b M(\varkappa_1 + \varkappa_2) + \nu^2,_b(L^2 + M^2)]]^{1/2}$$

where $L, M, N, _a = La + Mb; n,_b = Ma + Nb$.

Using the relations

$$\tilde{n}d\tilde{\Gamma} = (r + d)_\alpha \times (r + d)_\beta d\alpha d\beta = (1 + \nu^2 K)nd\Gamma + [(r_\alpha \times d)_\beta - (r_\beta \times d)_\alpha]d\alpha d\beta +$$

$$+ \nu O(\| \nu_\alpha \| + \| \nu_\beta \|)d\Gamma - (r_\alpha \times d)d\alpha d\beta = \frac{\nu}{|r_\beta|^2} r_\beta d\Gamma,$$

$$(r_\beta \times d)d\alpha d\beta = \frac{\nu}{|r_\alpha|^2} r_\alpha d\Gamma$$

$$(\ )_\alpha = r_\alpha \cdot \nabla(\ ), \quad (\ )_\beta = r_\beta \cdot \nabla(\ )$$

where K is the Gaussian of the curvature, and expansions of the form

$$\Sigma_r(x + \nu n, \xi) = \Sigma_r(x, \xi) + \Sigma_r,_n(x, \xi)\nu(x) + O(\|\nu^2\|)$$

we will linearize the singular part of the operator $\mathbf{S}$ the argument and the subscript r will be omitted for brevity. We obtain

$$\int_{\tilde{\Gamma}} \tilde{u} \cdot \Sigma \cdot \tilde{n}d\tilde{\Gamma} \cong \int_\Gamma \hat{u} \cdot (\Sigma + \Sigma,_n \nu) \cdot (nd\Gamma + [(r_\alpha \times d)_\beta - (r_\beta \times d)_\alpha]d\alpha d\beta) \cong$$

$$\cong \int_\Gamma \hat{u} \cdot (\Sigma \cdot n + \nabla\Sigma \odot (nn + aa + bb)\nu)d\Gamma +$$

$$+ \int_\Gamma \hat{u} \cdot [(\Sigma \cdot (r_\alpha \times d))_\beta - (\Sigma \cdot (r_\beta \times d))_\alpha]d\alpha d\beta =$$

$$= \int_\Gamma \hat{u} \cdot (\Sigma \cdot n + \nabla \cdot \Sigma\nu)d\Gamma - \int_\Gamma \hat{u}_\beta \cdot \Sigma \cdot (r_\alpha \times d) - \hat{u}_\alpha \cdot \Sigma \cdot (r_\beta \times d))d\alpha d\beta =$$

$$= \int_\Gamma \hat{u} \cdot \Sigma \cdot nd\Gamma + \int_\Gamma (-\rho\omega^2 \hat{u} \cdot U + \hat{u},_\alpha \cdot \Sigma \cdot a + \hat{u},_b \cdot \Sigma \cdot b)\nu d\Gamma \tag{9}$$

$$\hat{u}(x) = \tilde{u}(x + \nu(x)n(x))$$

We have used the formula of integration by parts and the equations of motion $\nabla \cdot \Sigma = -\rho\omega 2U$, $x \neq \xi$.

To linearize the regular part of the operator $\mathbf{S}$ we will introduce the function $\hat{p}(x) = \tilde{p}(x + \nu)(x)n(x))J(x)$. We obtain

$$\int_{\tilde{\Gamma}} \tilde{p}\cdot U d\tilde{\Gamma} \equiv \int_{\Gamma} \hat{p}\cdot U d\Gamma + \int_{\Gamma} \hat{p}\cdot U,_{n} \nu d\Gamma \tag{10}$$

We will introduce variations of the boundary fields by means of the formulae

$$\delta u = \hat{u} - u, \quad \delta p = \hat{p} - p, \quad x \in \Gamma$$

We substitute (9) and (10) into (8). Apart from small higher-orders with respect to the quantities in (6), we obtain the following linearized integral representation

$$\tilde{u}(\xi) - u(\xi) \equiv \mathbf{S}[\delta u, \ \delta p, \ \Gamma, \ \xi] + \mathbf{L}[\nu, \ u, \ p, \ \Gamma, \ \xi] + \mathbf{L}[\nu, \ \delta u, \ \delta p, \ \Gamma, \ \xi \tag{11}$$

$$\xi \in \tilde{V} \cap V$$

$$\mathbf{L} = (\mathbf{L}_1, \ \mathbf{L}_2, \ \mathbf{L}_3), \quad \mathbf{L}_r(\nu, \ u, \ p, \ \Gamma, \ \xi] = \int_{\Gamma} G_r(x, \ \xi)\nu(x) d\Gamma_x$$

$$G_r = \rho\omega^2 u \cdot U_r + p \cdot U_r,_{n} - u,_{a} \cdot \Sigma_r \cdot a - u,_{b} \cdot \Sigma_r \cdot b$$

It follows from the uniqueness of the solution of boundary-value problem (1)–(4) that $\|\delta\| \to 0$ and $\|\delta p\| \to 0$ as $\|\nu\| \to 0$. We will neglect the last term on the right-hand side of (11).

Consider the region $\Delta\Gamma$—a neighbourhood of $\partial(\tilde{V} \cap V)$. We specify a pair of points ξ and $\tilde{\xi}$ in a system of coordinates connected with Γ

$$\xi = x + \tau n(x), \quad \tilde{\xi} = x + (\tau + \nu(x))n(x), \quad \xi, \ \tilde{\xi} \in \Delta\Gamma$$

Note that if $\xi \to x \in \Gamma$, then $\tilde{\xi} \to \tilde{x} \in \tilde{\Gamma}$ and vice versa. At these points we determine the functions

$$\hat{u}(\xi) = \tilde{u}(\tilde{\xi}(\xi)) = \tilde{u}(\xi + \nu n), \quad \delta u(\xi) = \hat{u}(\xi) - u(\xi), \quad \xi \in \Delta\Gamma$$

We linearize the left-hand side of (11) in $\Delta\Gamma$ and take the limit as $\tau \to 0$. We obtain

$$\delta u(y) - u,_{n}(y)\nu(y) \equiv \lim_{\xi \to y}(\mathbf{S}[\delta u, \ \delta p, \ \Gamma, \ \xi] + \mathbf{L}[\nu, \ u, \ p, \ \Gamma, \ \xi]), \quad y \in \Gamma \tag{12}$$

We will use the idea of an integral in the sense of the principal Cauchy value and represent the limits on the right-hand side of (12) in terms of singular integral and their sudden changes. We will choose for a sudden change notation which emphasizes its dependence on an integrand of the form $\upsilon(x) \cdot A(x, \xi)$

$$\mathbf{M}_\gamma[\upsilon \cdot A, \ \Gamma, \ y], \quad y \in \Gamma$$

where γ is the angle between the direction of the trend of (ξ, y) and n, and υ satisfies the Hölder conditions.

Note the following simple property of the jumps

$$\mathbf{M}_\gamma[\upsilon \cdot A,_{a}, \ \Gamma, \ y], = -\mathbf{M}_\gamma[\upsilon,_{a} \cdot A, \ \Gamma, \ y] \tag{13}$$

and similarly for b.

The jumps of the Somigliana operator, due to the kernels U_r and P_r, are well known [6] and have the form

$$\mathbf{M}_\gamma[\upsilon \cdot U_r, \ \Gamma, \ y] = 0, \quad \mathbf{M}_\gamma[-\upsilon \cdot P_r, \ \Gamma, \ y] = \tfrac{1}{2}\chi\upsilon_r(y) \tag{14}$$

$$\mathbf{M}_\gamma[\delta p \cdot U_r - \delta u \cdot P_r, \ \Gamma, \ y] = \tfrac{1}{2}\chi\delta u_r(y), \quad \chi = \operatorname{sgn}(\sin\gamma) \tag{15}$$

To calculate the jumps in the operator $\mathbf{L}$ we will specify another representation of the kernel G_r. Using the relations

$$(\Sigma_r \cdot \nabla u) \odot (aa + bb + nn) = \Sigma_r \odot \nabla u$$

$$(\sigma \cdot \nabla U_r) \odot (aa + bb + nn) = \sigma \odot \nabla U_r$$

and the reciprocity theorem

$$\Sigma_r \odot \nabla u = \sigma \odot \nabla U_r$$

we will have

$$G_r = \rho\omega^2 u \cdot U_r + u,_n \cdot P_r - U_r,_a \cdot \sigma \cdot a - U_r,_b \cdot \sigma \cdot b$$

Using (13) and (14) we obtain

$$\mathbf{M}_\gamma[G_r \mathrm{v},\ \Gamma,\ y] = -\tfrac{1}{2}\chi u_r,_n(y)\mathrm{v}(y) \tag{16}$$

We replace the approximate equation in (12) by the rigorous equation and take (15) and (16) into account. We obtain

$$\tfrac{1}{2}(\delta u(y) - u,_n(y)\mathrm{v}(y)) = \mathbf{S}[\delta u,\ \delta p,\ \Gamma,\ y] + \mathbf{L}[\mathrm{v},\ u,\ p,\ \Gamma,\ y],\quad y \in \Gamma \tag{17}$$

We will restore the superscript $m = 0$ and 1 in (17). In view of the conjugation conditions (4), corresponding conjugation conditions hold for the variation of the boundary fields on Γ^-

$$\delta u^{(0)} = \delta u^{(1)},\quad \delta p^{(0)} = \delta p^{(1)},\quad x \in \Gamma^- \tag{18}$$

Using (4) and (18) we introduce the functions

$$u = u^{(0)} = u^{(1)},\quad p = p^{(0)} = p^{(1)},\quad x \in \Gamma^-$$

$$\delta u = \delta u^{(0)} = \delta u^{(1)},\quad \delta p = \delta p^{(0)} = \delta p^{(1)},\quad x \in \Gamma^-$$

Taking the above notation into account we rewrite (17) in the form of the system

$$\begin{gathered}\mathbf{S}^{(0)}[\delta u^{(0)},\ \delta p^{(0)},\ \Gamma^+,\ y] + \mathbf{L}^{(0)}[\mathrm{v},\ u^{(0)},\ p^{(0)},\ \Gamma^+,\ y] - \mathbf{S}^{(0)}[\delta u,\ \delta p,\ \Gamma^-,\ y] - \\ -\mathbf{L}^{(0)}[\mathrm{v},\ u,\ p,\ \Gamma^-,\ y] = \begin{cases} \tfrac{1}{2}(\delta u^{(0)}(y) - u,_n^{(0)}(y)\mathrm{v}(y)), & y \in \Gamma^+ \\ \tfrac{1}{2}(\delta u(y) - u,_n^{(0)}(y)\mathrm{v}(y)), & y \in \Gamma^- \end{cases} \\ \mathbf{S}^{(1)}[\delta u,\ \delta p,\ \Gamma^-,\ y] + \mathbf{L}^{(1)}[\mathrm{v},\ u,\ p,\ \Gamma^-,\ y] = \tfrac{1}{2}(\delta u(y) - u,_n^{(1)}(y)\mathrm{v}(y)),\quad y \in \Gamma^-\end{gathered} \tag{19}$$

By virtue of the boundary conditions (3) the variations of the boundary fields satisfy the following boundary conditions on Γ^+

$$\delta u^{(0)} = 0,\quad x \in \Gamma_u^+;\quad \delta p^{(0)} = 0,\quad x \in \Gamma_p^+ \cup \Gamma^* \tag{20}$$

and we can introduce the following boundary conditions δs on Γ^+

$$\delta s = \begin{cases} \delta p^{(0)}, & x \in \Gamma_u^+ \\ \delta u^{(0)}, & x \in \Gamma_p^+ \cup \Gamma^* \end{cases}$$

The linearized system of boundary integral equations (19), taking (20) into account relates the variations of the boundary fields δs, δu, δp to the variations in the shape of the boundaries v. When solving the direct problem with specified shapes of the boundaries Γ^+ and Γ^-, we can specify the function v and calculate the variations of the boundary fields. In this case the system obtained represents three vector complex linear equations in the three vector complex functions. Knowing the variations of the boundary fields we can calculate the differences of the displacement fields at any internal point of the body from (11).

The special boundary integral equations (19) are of greatest interest when solving geometrical inverse problems, when it is required to establish an unknown boundary surface (or part of it), which we will denote by $\tilde{\Lambda}$, $\tilde{\Lambda} \subset$ (on on $\tilde{\Gamma}^- \cup \tilde{\Gamma}^+$) on the basis of information on the wave field on a part T of the surface $\tilde{\Gamma}^+$ accessible to observation. Hence, the formulation of the inverse boundary-value problem requires an additional boundary condition, namely

$$\tilde{u} = f, \quad x \in T \tag{21}$$

If we assume that a priori information is available on the required surface $\tilde{\Lambda}$, expressed in the form of the known surface

$$\Lambda \subset (\Gamma^- \cup \Gamma^+)(\nu(x) = 0, \quad (\Gamma^- \cup \Gamma^+) \setminus \Lambda = (\tilde{\Gamma}^- \cup \tilde{\Gamma}^+) \setminus \tilde{\Lambda})$$

which is close to the required surface in the sense of (6), a search for the variation of the shape of the known surface is also the aim of the geometrical inverse problem. We will call this problem the linearized geometrical inverse problem. After solving the direct problem with a known shape of the boundaries Γ^- and Γ^+, the system of special boundary integral equations (19), using the additional boundary condition (21), is a system of linear integral equations in the unknown variations of the boundary fields and the variation of the shape, i.e. it solves the linearized geometrical inverse problem.

We can suggest an approach to solving nonlinear geometrical inverse problems based on the use of a priori information on the shape of the required surface $\tilde{\Lambda}$, which is expressed by specifying the surface $\Lambda^{(0)}$ (not necessarily close to the required surface). Beginning with $\Lambda^{(0)}$, we construct a recurrent sequence of surfaces $\Lambda^{(0)}, \Lambda^{(1)}, \Lambda^{(2)}, \ldots$, which "contracts" to the required shape of the surface if $\Lambda^{(0)}$ is correctly chosen. To transfer from $\Lambda^{(i)}$ to $\Lambda^{(i+1)}$ it is sufficient to solve the linearized geometrical inverse problem and use representation (7) [2].

We will consider some particular examples of the formulation of special boundary integral equations. In all cases we will assume that the initial boundary condition is formulated on the part of the boundary of the body ($T \subset \tilde{\Gamma}_p^+$) that is stress-free.

Problem 1. Suppose an elastic uniform medium occupies a spatially simply connected volume $\tilde{W}$ (there is no defect). It is required to determine the shape of the clamped part of the boundary $\tilde{\Gamma}^+(\tilde{\Lambda} = \tilde{\Gamma}_u^+)$ on the basis of information on the wave field of the displacement of the part of the boundary $T = \tilde{\Gamma}^+$ ($= \Gamma^+$) that is stress-free.

We use the top equation in (19) and, omitting the superscript $m = 0$, we obtain

$$\int_{\Lambda(=\Gamma_u^+)} (\delta p(x) \cdot U_r(x, y) + \nu(x) p(x) \cdot U_{r,n}(x, y)) d\Lambda -$$
$$- \int_{\Gamma^*} \delta u(x) \cdot P_r(x, y) d\Gamma^* = \int_{T(=\Gamma_p^+)} (f(x) - u(x)) \cdot P_r(x, y) d\Gamma_p^+ +$$
$$+ \begin{cases} -\tfrac{1}{2} u_{r,n}(y) \nu(y), & y \in \Gamma_u^+, \\ \tfrac{1}{2} \delta u_r(y), & y \in \Gamma^* \\ \tfrac{1}{2}(f_r(y) - u_r(y)), & y \in \Gamma_p^+ \end{cases} \tag{22}$$

Problem 2. Suppose, as in the previous problem, that there is no defect. It is required to determine the part of the boundary $\tilde{\Gamma}^{-+}(\tilde{\Lambda} \subset \tilde{\Gamma}_p^+)$ that is stress-free on the basis of information on the wave field of the displacements on the known part of the free surface $T = \tilde{\Gamma}_p{}^+ \setminus \tilde{\Lambda}$ ($= \tilde{\Gamma}_p \setminus \Lambda$).

Using the top equation in (19) and omitting the superscript $m = 0$ we obtain

$$\int_{\Lambda(\subset\Gamma_p^+)} (\delta u(x) \cdot P_r(x, y) + \nu(x) G_r(x, y)|_{p=0}) d\Lambda -$$
$$- \int_{\Gamma^*} \delta u(x) \cdot P_r(x, y) d\Gamma^* + \int_{\Gamma_u^+} \delta p(x) \cdot U_r(x, y) d\Gamma_u^+ =$$
$$= \int_{T(=\Gamma_p^+ \setminus \Lambda)} (f(x) - u(x)) \cdot P_r(x, y) d\Gamma_p^+ +$$
$$+ \begin{cases} 0, & y \in \Gamma_u^+ \\ \tfrac{1}{2} \delta u_r(y), & y \in \Gamma^* \\ \tfrac{1}{2}(\delta u_r(y) - u_{r,n}(y) \nu(y)) & y \in \Lambda \\ \tfrac{1}{2}(f_r(y) - u_r(y)), & y \in \Gamma_p^+ \setminus \Lambda \end{cases} \tag{23}$$

Equations (22) and (23) can be simplified if we get rid of the additional unknowns by a special choice of the fundamental solutions. We will assume that the fundamental solutions satisfy Eq. (5) in W and the following boundary conditions

$$U_r = 0, \quad x \in \Gamma_u^+; \quad P_r = 0, \quad x \in \Gamma_p^+ \cup \Gamma^* \tag{24}$$

Since the special fundamental solutions in W differ from the fundamental solutions $\mathbf{R}^3$ (5) on a function that is regular in W, they retain the limit properties of the latter and (22) takes the form

$$\int_{\Lambda(=\Gamma_u^+)} \nu(x) u,_n(x) \cdot P_r(x, y) d\Lambda = \begin{cases} -\tfrac{1}{2} u_r,_n(y)\nu(y), & y \in \Gamma_u^+ \\ f_r(y) - u_r(y), & y \in \Gamma_p^+ \end{cases}$$

Note that in this case $p \cdot U_r,_n = u,_n \cdot P_r$. Using the special fundamental solutions (24) we can convert system (23) to the following system

$$\int_{\Lambda(\subset\Gamma_p^+)} \nu(x) G_r(x, y)\Big|_{p=0} d\Lambda = \begin{cases} 0, & y \in \Gamma_u^+ \\ f_r(y) - u_r(y), & y \in \Gamma_p^+ \setminus \Lambda \end{cases}$$

The construction of special fundamental solutions, which satisfy boundary conditions (24), is an independent problem, to solve which we need to formulate the corresponding boundary integral equations. Moreover, as a result of the iterative procedure of successive approximations, $\Lambda^{(i)}$ change, which involves recalculating the special fundamental solutions at each iteration. A compromise approach is better, namely, from the fundamental solutions the boundary conditions only need to be satisfied on a known part of the boundary $\tilde{\Gamma}^+ \setminus \tilde{\Lambda}$.

Problem 3. It is required to reconstruct the shape of an elastic inclusion $\tilde{\Gamma}^-$ in an elastic body $\tilde{W}$ on the basis of information on the wave field of displacements on the part of the surface $\Gamma^+ (T \subset \Gamma_p^+)$ that is stress-free. We will use the compromise version of choosing the fundamental solutions, when $U_r^{(0)}$ satisfy the boundary conditions only on the known external boundary Γ^+, i.e. (24).

In this case system (19) takes the form

$$-\mathbf{S}^{(0)}[\delta u, \delta p, \Gamma^-, y] - \mathbf{L}^{(0)}[\nu, u, p, \Gamma^-, y] = \begin{cases} 0, & y \in \Gamma_u^+ \\ f(y) - u(y), & y \in T \end{cases} \tag{25}$$

$$-\mathbf{S}^{(m)}[\delta u, \delta p, \Gamma^-, y] - \mathbf{L}^{(m)}[\nu, u, p, \Gamma^-, y] =$$

$$= \tfrac{1}{2}(\delta u(y) - u,_n^{(m)}(y)\nu(y)), \quad y \in \Gamma^-, \quad m = 0, 1$$

Note that in the case of an absolutely rigid inclusion or a cavity, in system (25) it is sufficient to put u and δu or p and δp, respectively, equal to zero; here it is not necessary to use the lower equation in (25) when $m = 1$.

Notes 1. The linearized boundary integral equations obtained retain the singular order singularities present in the usual boundary integral equations.

2. The solution of the above systems is an ill-posed problem. Together with these equations we can consider similar ones, but compiled for different frequencies ω and different positions of the external load Γ^*. Here, the unknown variations of the boundary fields will change, while the variations in the shape remain constant. Such an overdetermination is a favourable factor when solving ill-posed problems.

3. Reformulation of the equations obtained for elastic media in the case of acoustic media leads to considerable simplifications. As far as the physical meaning of the wave fields is concerned, the Somigliana operator becomes the Helmholtz–Kirchhoff operator, and the kernel G of the operator $\mathbf{L}$ takes the form

$$G = k^2 uU + u,_n U,_n - u,_a U,_a - u,_b U,_b$$

where k is the wave number and U is the corresponding acoustic potential. A cavity corresponds to an acoustically rigid surface, while an absolutely rigid inclusion corresponds to an acoustically soft surface.

4. The technique described for constructing special boundary integral equations can be extended to the case when the volume W is semi-bounded. In particular, the antiplane problem for an elastic half-space with a cylindrical inclusion was considered in [4, 5]; the special boundary integral equations obtained earlier (to reconstruct the shape

of the section of the inclusion boundary) are a special case of Eqs (25). Here the special fundamental solutions, which satisfy the boundary condition on the boundary of the half-space, have an analytical representation in terms of Hankel functions. The results of numerical experiments, which demonstrate the practicability of the proposed method of solving geometrical inverse problems, were given in [4, 5].

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